

Hyperfinite Lévy Processes

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Abstract

A hyperfinite Lévy process is an infinitesimal random walk (in the sense of nonstandard analysis) which with probability one is finite for all finite times. We develop the basic theory for hyperfinite Lévy processes and find a characterization in terms of transition probabilities. The standard part of a hyperfinite Lévy process is a (standard) Lévy process, and we show that given a generating triplet (γ, C, ν) for standard Lévy processes, we can construct hyperfinite Lévy processes whose standard parts correspond to this triplet. Hence all Lévy laws can be obtained from hyperfinite Lévy processes. The paper ends with a brief look at Malliavin calculus for hyperfinite Lévy processes including a version of the Clark-Haussmann-Ocone formula.

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Intuitively, Lévy processes are just continuous time analogues of random walks with independent and stationary increments. The purpose of the present paper is to make this intuition precise by studying infinitesimal random walks (in the sense of nonstandard analysis) and show that they correspond exactly to (standard) Lévy processes. In the founding paper of nonstandard stochastic analysis [3], R.M. Anderson showed that a Bernoulli random walk with infinitesimal time steps generates (standard) Brownian motion, and this paper may be regarded as an extension of Anderson's study to infinitesimal random walks in general.

In our presentation, we start with the random walks and use them to generate Lévy processes. S. Albeverio and F.S. Herzberg [2] have studied the opposite situation where the Lévy processes are the initially given objects, and where the random walks are constructed from the Lévy processes. The two papers have very little in common, but where they overlap, priority belongs to Albeverio and Herzberg.

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The paper is organized as follows. In the first section, we define a hyperfinite Lévy process as a hyperfinite random walk in ${}^*\mathbf{R}^d$ which (with probability one) stays finite for all finite times, and we prove some simple but useful identities. In section 2, we show that hyperfinite Lévy processes with limited increments are S-integrable of all orders, and in the following section we show that all hyperfinite Lévy processes can be approximated by processes with limited increments. Section 4 contains a characterization of hyperfinite Lévy processes in terms of transition probabilities, and in section 5 we show that hyperfinite Lévy processes can be decomposed into a diffusion part and a pure jump part in a natural way. We then turn to the relationship between hyperfinite and standard Lévy processes, and prove (in section 6) that the standard part of a hyperfinite Lévy process is a standard Lévy process. In order to understand this relationship better, we introduce Lévy measures and covariance matrices from a nonstandard perspective in section 7, and then prove a nonstandard version of the Lévy-Khintchine formula in section 8. Using this formula, we show in section 9 that “all” standard Lévy processes can be obtained as standard parts of hyperfinite Lévy processes, where “all” means that given a generating triplet (γ, C, ν) for Lévy processes, we can in a constructive way find a random walk with standard part corresponding to this triplet. Since two Lévy processes with the same triplet have the same law, this also means that all Lévy laws can be obtained from hyperfinite Lévy processes. We end the paper by taking a brief and informal look at Malliavin calculus with respect to hyperfinite Lévy processes.

I shall assume that the reader is familiar with the basic results of nonstandard probability theory, and I shall use the notation and terminology of the book by Albeverio et al. [1] and the survey paper by Lindstrøm [24]. Some of the results that I shall use from nonstandard martingale theory, can only be found in full generality in the original papers by Lindstrøm [24] and Hoover and Perkins [19], but they usually generalize easily from the results in [1] and [24]. Formally, no previous knowledge of (standard) Lévy processes is required, but it may be an advantage to take a look at the books by Bertoin [6] and Sato [34], and the interesting collection [4] edited by Barndorff-Nielsen, Mikosch and Resnick. For readers who just want a quick introduction to the basic ideas, the first chapter of Protter’s book [33] is excellent.

1 Basic definitions

To describe our random walks, we first introduce a hyperfinite timeline $T = \{k\Delta t : k \in {}^*\mathbf{N}_0\}$, where Δt is infinitesimal, and where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. We shall be considering internal processes $X : \Omega \times T \rightarrow {}^*\mathbf{R}^d$ where (Ω, \mathcal{F}, P) is an internal probability space. Following standard notation, we write $\Delta X(t) := X(t + \Delta t) - X(t)$ for the forward increment of X at a time $t \in T$. The (completed) Loeb space of (Ω, \mathcal{F}, P) will be denoted by $(\Omega, \mathcal{F}_L, P_L)$.

Our hyperfinite random walks will be specified by a hyperfinite set $A = \{a_1, a_2, \dots, a_H\}$ of elements in ${}^*\mathbf{R}^d$ and a set of positive numbers $\{p_a\}_{a \in A}$ in

${}^*\mathbf{R}$ such that $\sum_{a \in A} p_a = 1$. We call A the set of *increments* and $\{p_a\}_{a \in A}$ the *transition probabilities*.

Definition 1.1 A hyperfinite random walk with increments A and transition probabilities $\{p_a\}_{a \in A}$ is an internal process $L : \Omega \times T \rightarrow {}^*\mathbf{R}^d$ such that:

- (i) $L(0) = 0$.
- (ii) The increments $\Delta L(0), \Delta L(\Delta t), \dots, \Delta L(t), \dots$ are * -independent.
- (iii) All increments $\Delta L(t)$ have the distribution specified by A and $\{p_a\}_{a \in A}$, i.e.

$$P[\Delta L(\omega, t) = a] = p_a$$

for all $t \in T$ and all $a \in A$.

Given a hyperfinite random walk L , we shall let $\{\mathcal{F}_t\}_{t \in T}$ be the internal filtration generated by L .

Before we continue, it may be useful to take a look at three simple examples. As will become clear later, these examples basically sum up the three “typical behaviors” of hyperfinite Lévy processes: deterministic drift, (martingale) diffusion, and jumps (compare the decomposition in Theorem 5.1).

Example 1: All three processes take values in ${}^*\mathbf{R}$:

- a) Choose a real number α , let $A = \{\alpha \Delta t\}$ and $p_{\alpha \Delta t} = 1$. Then L is the deterministic motion $L(\omega, t) = \alpha t$.
- b) Let $A = \{-\sqrt{\Delta t}, \sqrt{\Delta t}\}$ and put $p_{-\sqrt{\Delta t}} = p_{\sqrt{\Delta t}} = \frac{1}{2}$. Then L is Anderson’s random walk [3].
- c) Let ν be a real number, let $A = \{0, 1\}$ and put $p_0 = 1 - \nu \Delta t$, $p_1 = \nu \Delta t$. Then L is Loeb’s Poisson process [26]. ♠

We now introduce the vector $\mu_L \in {}^*\mathbf{R}^d$ by

$$\mu_L := \frac{1}{\Delta t} E[\Delta L(0)] = \frac{1}{\Delta t} \sum_{a \in A} a p_a$$

and note that

$$E[L(t)] = E\left[\sum_{s < t} \Delta L(s)\right] = \frac{t}{\Delta t} E[\Delta L(0)] = \mu_L t$$

Observe that the process $M_L(t) := L(t) - \mu_L t$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \in T}$ generated by L . We also introduce a nonnegative number $\sigma_L \in {}^*\mathbf{R}^d$ by

$$\sigma_L^2 := \frac{1}{\Delta t} E[|\Delta L(0)|^2] = \frac{1}{\Delta t} \sum_{a \in A} |a|^2 p_a$$

and note the following simple, but useful identity:

Lemma 1.2 For all $t \in T$

$$E[|L(t)|^2] = \sigma_L^2 t + |\mu_L|^2 t(t - \Delta t)$$

Proof: Since $\Delta L(s)$ and $\Delta L(r)$ are independent for $s \neq r$, we get

$$\begin{aligned} E[|L(t)|^2] &= E\left[\left(\sum_{r < t} \Delta L(r)\right) \cdot \left(\sum_{s < t} \Delta L(s)\right)\right] = \\ &= \sum_{s < t} E[|\Delta L(s)|^2] + \sum_{0 \leq s \neq r < t} E[\Delta L(r) \cdot \Delta L(s)] = \\ &= \frac{t}{\Delta t} \sigma_L^2 \Delta t + \sum_{0 \leq r \neq s < t} |\mu_L|^2 \Delta t^2 = \sigma_L^2 t + |\mu_L|^2 t(t - \Delta t) \quad \spadesuit \end{aligned}$$

So far we have not put any size restrictions on our process L . As we want to turn L into a standard process by taking standard parts, the weakest size restriction that seems reasonable, is the following:

Definition 1.3 *Let L be a hyperfinite random walk. We call L a hyperfinite Lévy process if the set*

$$\{\omega \mid L(\omega, t) \text{ is finite for all finite } t \in T\}$$

has Loeb measure 1.

At first glance this definition may seem impractical as there is no obvious way to check that it is satisfied, but as we shall see in Theorem 4.3, it is possible to find descriptions in terms of A and p_a that are easy to check. To find these descriptions, we first need some simple estimates.

2 Hyperfinite Lévy processes with limited increments

In this section, we shall prove a basic estimate that will give us much better control over our hyperfinite Lévy processes. We begin with a lemma that is well known, but which seems difficult to find in the literature in its most general form. Let us write $q << p$ if $q < p - \epsilon$ for all infinitesimal ϵ :

Lemma 2.1 *Assume that (Ω, \mathcal{A}, P) is an internal measure space such that $P(\Omega)$ is finite, and let $F : \Omega \rightarrow^* \mathbf{R}$ be an \mathcal{A} -measurable internal function. Assume that $\int |F|^p dP$ is finite for some finite $p \in {}^*\mathbf{R}_+$. Then $|F|^q$ is S -integrable for all $q \in {}^*\mathbf{R}_+$, $q << p$.*

Proof: Since $p > q$ and $\int |F|^p dP$ is finite, $\int |F|^q dP$ must also be finite. Hence it suffices to show that if $A \in \mathcal{A}$, $P(A) \approx 0$, then $\int_A |F|^q dP \approx 0$. By Hölder's inequality

$$\int_A |F|^q dP = \int 1_A |F|^q dP \leq \left(\int 1_A^{\frac{p}{p-q}} dP \right)^{\frac{p-q}{p}} \left(\int |F|^p dP \right)^{\frac{q}{p}} =$$

$$= P(A)^{\frac{p-q}{p}} \left(\int |F|^p dP \right)^{\frac{q}{p}}$$

which is infinitesimal since $P(A) \approx 0$ and $\frac{p-q}{p}$ is noninfinitesimal. ♠

Our estimate only applies to processes with jumps that are not too big:

Definition 2.2 *A hyperfinite Lévy process has limited increments if the increments are S -bounded, i.e., there is an $N \in \mathbf{N}$ such that $|a| \leq N$ for all $a \in A$.*

We are now ready for the basic estimate. It (and its proof) is based on a similar result for standard Lévy processes (see, e.g., Protter [33, Chapter 1, Theorem 34]).

Theorem 2.3 *Let L be a hyperfinite Lévy process with limited increments. Then $|L_t|^p$ is S -integrable for all finite $p \in {}^*\mathbf{R}_+$ and all finite $t \in T$.*

Proof: If $L \equiv 0$, there is nothing to prove. Assuming that $L \not\equiv 0$, the stopping time

$$\tau_K = \min\{t \in T : |L_t| \geq K\}$$

(putting $\tau_K = \infty$ if such a t does not exist) is well-defined and different from ∞ a.s. for any positive $K \in {}^*\mathbf{R}$. Note that if K is infinite, then clearly $\tau_K > 1$ almost everywhere (as a matter of fact, τ_K is infinitely large almost everywhere!). In particular,

$$P\{\tau_K > 1\} > \frac{1}{2}$$

holds for all infinite K . By “underflow”, it must also hold for all sufficiently large, finite K . Fix such a finite K , and make sure that it is noninfinitesimal and larger than all the jumps of L . For later use, we define

$$\alpha = E[e^{-\tau_K}]$$

and observe that by our choice of K , $\alpha < 1$.

We now define a sequence of stopping times $\{\sigma_n\}$ by letting $\sigma_1 = \tau_K$ and putting

$$\sigma_n = \min\{t \in T : |L_t - L_{\sigma_{n-1}}| \geq K\}$$

Observe that all the increments $\sigma_n - \sigma_{n-1}$ are independent and have the same distribution as τ_K . Hence

$$E[e^{-\sigma_n}] = E[e^{-\tau_K}]^n = \alpha^n$$

Since K is larger than all the increments of L , we have $|L_{\sigma_k} - L_{\sigma_{k-1}}| < 2K$. Hence

$$P[|L_t| \geq 2nK] \leq P[\sigma_n < t] \leq \frac{E[e^{-\sigma_n}]}{e^{-t}} \leq e^t \alpha^n$$

If we choose $\epsilon \in \mathbf{R}_+$ so small that ${}^\circ(\alpha e^{2K\epsilon}) < 1$, we have for any finite t

$$\begin{aligned} E \left[e^{\epsilon |L_t|} \right] &= \sum_{n \in {}^*\mathbf{N}} \int_{[2(n-1) \leq |L_t| < 2nK]} e^{\epsilon |L_t|} dP \leq \\ &\leq \sum_{n \in {}^*\mathbf{N}} e^{\epsilon 2nK} \cdot e^t \cdot \alpha^{n-1} = e^t e^{2K\epsilon} \sum_{n \in {}^*\mathbf{N}} (e^{2K\epsilon} \alpha)^{n-1} < \infty \end{aligned}$$

Since $e^{\epsilon |L_t|} > |L_t|^p$ when $|L_t|$ is large, it follows that $E(|L_t|^p)$ is finite for all finite $p \in {}^*\mathbf{R}_+$. The S-integrability follows from the lemma. ♠

It is important to realize that the theorem above only applies to processes with limited increments. It is not difficult to construct hyperfinite Lévy processes with unlimited increments that have infinite expectations. Here is one such example:

Example 2 Pick an infinite $N \in {}^*\mathbf{N}$ and let $A = \{0, 1, 2, \dots, N\}$. For $n > 0$, put $p_n = \frac{\Delta t}{n^2}$, and let $p_0 = 1 - \Delta t \sum_{n=1}^N \frac{1}{n^2}$. Observe that

$$E[L(t)] = \frac{t}{\Delta t} E[\Delta L(0)] = \frac{t}{\Delta t} \sum_{n=1}^N n \frac{\Delta t}{n^2} = t \sum_{n=1}^N \frac{1}{n}$$

which is infinite for noninfinitesimal t . However, using a little combinatorics, it is not hard to see that $L(t)$ is finite P_L -almost everywhere (I do not give the proof as the statement will follow immediately from Theorem 4.3). Hence L is a nonintegrable hyperfinite Lévy process. ♠

We shall take a look at two very useful corollaries of the theorem above. The first gives a characterization of when a hyperfinite random walk with limited increments is a hyperfinite Lévy process. Recall the quantities

$$\mu_L := \frac{1}{\Delta t} E[\Delta L(0)] = \frac{1}{\Delta t} \sum_{a \in A} a p_a$$

and

$$\sigma_L^2 := \frac{1}{\Delta t} E[|\Delta L(0)|^2] = \frac{1}{\Delta t} \sum_{a \in A} |a|^2 p_a$$

in the previous section.

Corollary 2.4 *Let L be a hyperfinite random walk with limited increments. Then L is a hyperfinite Lévy process if and only if μ_L and σ_L are finite.*

Proof: Assume first that L is an hyperfinite Lévy process. Then according to the theorem

$$\mu_L = E[L(1)]$$

is finite. By Lemma 1.2,

$$E[|L(t)|^2] = \sigma_L^2 t + |\mu_L|^2 t(t - \Delta t)$$

and since the theorem tells us that the left hand side is finite for all finite t , σ_L is finite.

For the converse, we observe that if μ_L and σ_L are finite, then

$$E[|L(t)|^2] = \sigma_L^2 t + |\mu_L|^2 t(t - \Delta t)$$

is finite for all finite t . Moreover, $M_L(t) = L(t) - \mu_L t$ is a martingale which is square integrable in the sense that $E[M_L(t)^2]$ is finite for all finite t (in the terminology of [1] and [24], M_L is a λ^2 -martingale). It is well-known that almost all paths of such martingales are finite for all finite t (see, e.g., [1, page 119] or [24, Prop. 7.2]). Since $L(t) = \mu_L t + M_L(t)$ where μ_L is finite, the same obviously applies to L . ♠

As already mentioned, we shall later (section 4) find a related (but somewhat more complicated) characterization of when a general hyperfinite random walk is a hyperfinite Lévy process.

Our second corollary just sums up what we already know about the decomposition of L into a drift part and a martingale part.

Corollary 2.5 *A hyperfinite Lévy process L with limited increments can be decomposed as*

$$L(t) = \mu_L t + M_L(t)$$

where $\mu_L \in {}^*\mathbf{R}^d$ is finite and M_L is a martingale such that $|M_L(t)|^p$ is S -integrable for all finite t and all finite $p \in {}^*\mathbf{R}_+$. In particular, M_L is an SL^2 -martingale (in the terminology of [1], an SL^2 -martingale is just an internal martingale such that $|M_L(t)|^2$ is S -integrable for all finite t).

Since a lot is known about SL^2 -martingales, the corollary will be quite useful in proving path properties of hyperfinite Lévy processes. However, for this method to work efficiently, we need to know how well arbitrary hyperfinite Lévy processes can be approximated by processes with limited increments. This is the topic of the next section.

3 Approximating by processes with limited increments

In this section, we shall prove that hyperfinite Lévy processes can be approximated arbitrarily well by hyperfinite Lévy processes with limited increments. Introducing the notation

$$q_k = \frac{1}{\Delta t} \sum_{|a| > k} p_a$$

for any positive $k \in {}^*\mathbf{R}$, we first prove a simple lemma that will also be useful in other contexts.

Lemma 3.1 Assume that L is a hyperfinite Lévy process. Then

$$\lim_{k \rightarrow \infty} {}^\circ q_k = 0$$

in the sense that for or any $\epsilon \in \mathbf{R}_+$, there is a $N \in \mathbf{N}$ such that $q_k < \epsilon$ whenever $k \geq N$.

Proof: If the lemma did not hold, there had to be a number $b \in \mathbf{R}_+$ such that $q_k > b$ for all finite $k \in {}^*\mathbf{R}_+$. By *overflow* there would then be an infinite K such that $q_K > b$. By simple combinatorics, this means that the probability of L making *no* jump of size K or larger before time 1, is less than

$$(1 - b\Delta t)^{1/\Delta t} \approx e^{-b} < 1$$

Hence with noninfinitesimal probability, L makes a jump of infinite size before time 1, which is absurd since almost all the paths of L are finite for all finite $t \in T$. ♠

For any positive $k \in {}^*\mathbf{R}$, let $L^{>k}$ and $L^{k\leq}$ be the “truncated” (from below and above) processes

$$L^{>k}(\omega, t) = \sum \{ \Delta L(\omega, s) : s < t \text{ and } |\Delta L(\omega, s)| > k \}$$

and

$$L^{k\leq}(\omega, t) = \sum \{ \Delta L(\omega, s) : s < t \text{ and } |\Delta L(\omega, s)| \leq k \}$$

Lemma 3.2 Assume that L is a hyperfinite Lévy process. For all sufficiently large, finite numbers $k \in {}^*\mathbf{R}$, the processes $L^{>k}$ and $L^{k\leq}$ are hyperfinite Lévy processes.

Proof: Since the processes obviously are hyperfinite random walks, we just have to check that they a.s. remain finite in finite time. Observe that since $L^{k\leq} = L - L^{>k}$, and the difference between two hyperfinite Lévy processes is itself a hyperfinite Lévy process, it suffices to show that $L^{>k}$ is a hyperfinite Lévy process for k sufficiently large.

The previous lemma tells us that if we choose k finite, but sufficiently large, $\alpha := q_k$ is finite. We shall first prove that for any finite $m > k$, the process

$$L^{(k,m]}(\omega, t) = \sum \{ \Delta L(\omega, s) : s < t \text{ and } k < |\Delta L(\omega, s)| \leq m \}$$

is a hyperfinite Lévy process. This is straightforward: since $L^{(k,m]}$ has limited increments, the process can only become infinite by making infinitely many jumps. Since the probability of L making a jump larger than k at any given time t , is $q_k \Delta t = \alpha \Delta t$, basic combinatorics show that the probability of $L^{(k,m]}$ making infinitely many jumps in finite time is zero. (To see this, note that the probability of L making exactly n jumps of size larger than k before time t is

$$\binom{t/\Delta t}{n} (1 - \alpha \Delta t)^{t/\Delta t - n} (\alpha \Delta t)^n \approx \left(\frac{t}{\Delta t} \right)^n \frac{1}{n!} e^{-\alpha t} (\alpha \Delta t)^n \approx e^{-\alpha t} \frac{(\alpha t)^n}{n!}$$

Summing over all finite n , we see that the (Loeb)-probability of $L^{(k,m]}$ making just a finite number of jumps is one.)

We now turn to the original process $L^{>k}$. If this is not a hyperfinite Lévy process, there must be a finite t such that

$$p := P_L[L^{>k}(s) \text{ is infinite for some } s \leq t]$$

is noninfinitesimal. Combinatorics tell us that

$$P[\omega : L^{(k,m]}(\omega, s) = L^{>k}(\omega, s) \text{ for all } s \leq t] = (1 - q_m \Delta t)^{t/\Delta t} \approx e^{-q_m t}$$

According to the lemma, we can get $e^{-q_m t}$ as close to 1 as we want by choosing m sufficiently large (but finite). In particular, we can get $1 - e^{-q_m t} < p$. But then $L^{>k}$ equals the a.s. finite process $L^{(k,m]}$ on a set of measure larger than $1 - p$, and this is a contradiction. ♠

Remark As we shall see in the next section, the lemma actually holds for *all* noninfinitesimal k (but not, in general, for infinitesimal k).

We can use essentially the same argument to prove the result we have been aiming at:

Proposition 3.3 *Let L be a hyperfinite Lévy process. For each finite $t \in T$ and each $\epsilon \in \mathbf{R}_+$, there is a hyperfinite Lévy process \hat{L} with limited increments such that*

$$P[\omega : L(\omega, s) = \hat{L}(\omega, s) \text{ for all } s \leq t] > 1 - \epsilon$$

Proof: We know that for all sufficiently large $k \in \mathbf{R}_+$, the process $L^{\leq k}$ is a hyperfinite Lévy process with limited increments. Simple combinatorics tells us that

$$P[\omega : L(\omega, s) = L^{\leq k}(\omega, s) \text{ for all } s \leq t] = (1 - q_k \Delta t)^{t/\Delta t} \approx e^{-q_k t}$$

According to the lemma, $e^{-q_k t} \rightarrow 0$ as $k \rightarrow \infty$, and hence we can put $\hat{L} = L^{\leq k}$ for a sufficiently large k . ♠

The proposition above is useful in proving path properties of hyperfinite Lévy processes as it allows us to reduce the problem to processes with limited increments. We shall see examples of this technique in later sections.

4 A characterization of hyperfinite Lévy processes

Let L be a hyperfinite random walk. How can we tell from the increments a and the transition probabilities p_a whether or not L is a hyperfinite Lévy process?

We know from the previous section that if L is a hyperfinite Lévy process, then for k finite and sufficiently large,

$$q_k = \frac{1}{\Delta t} \sum_{|a| > k} p_a$$

is finite. To take a closer look at the distribution of the noninfinitesimal increments in A , we introduce an internal measure on all internal subset B of ${}^*\mathbf{R}^d$ by

$$\hat{\nu}(B) = \frac{1}{\Delta t} \sum_{a \in B} p_a$$

Note that $\hat{\nu}(B)$ is a natural generalization of q_k .

Proposition 4.1 *Let B be an internal subset of ${}^*\mathbf{R}^d$ which does not contain any infinitely small elements. Then $\hat{\nu}(B)$ is finite.*

Proof: We first observe that by Lemma 3.1 it suffices to show the proposition when B is bounded above by a real number k . By Lemma 3.2 we may also assume that the process

$$L^{\leq k}(\omega, t) = \sum \{ \Delta L(\omega, s) : s < t \text{ and } |\Delta L(\omega, s)| \leq k \}$$

is a hyperfinite Lévy process. Since $L^{\leq k}$ has limited increments,

$$\sigma_{L^{\leq k}}^2 = \frac{1}{\Delta t} \sum_{|a| \leq k} a^2 p_a$$

is finite by Corollary 2.4. Also, since B is internal and does not contain any infinitely small elements, there is a positive real number ϵ such that $\epsilon < |b|$ for all $b \in B$. We thus have

$$\epsilon^2 \hat{\nu}(B) = \epsilon^2 \frac{\sum_{a \in B} p_a}{\Delta t} < \frac{\sum_{a \in B} a^2 p_a}{\Delta t} \leq \sigma_{L^{\leq k}}^2$$

Since ϵ is noninfinitesimal and $\sigma_{L^{\leq k}}$ is finite, the lemma follows. \spadesuit

As a corollary, we may now extend Lemma 3.2. For any internal subset Λ of ${}^*\mathbf{R}^d$, we write

$$L^\Lambda(\omega, t) = \sum \{ \Delta L(\omega, s) : s < t \text{ and } \Delta L(\omega, s) \in \Lambda \}$$

Corollary 4.2 *Let Λ be an internal subset of ${}^*\mathbf{R}^d$. Assume that either Λ does not contain any infinitely small elements, or that Λ contains all infinitely small elements in ${}^*\mathbf{R}^d$. Then L^Λ is a hyperfinite Lévy process. In particular, the processes $L^{>k}$ and $L^{k\leq}$ in Lemma 3.2 are hyperfinite Lévy processes for all noninfinitesimal k .*

Proof: First observe that since $L^{\Lambda^c}(t) = L(t) - L^\Lambda(t)$ and the difference of two hyperfinite Lévy processes is itself a hyperfinite Lévy process, it suffices to prove the case where Λ does not contain any infinitely small elements in ${}^*\mathbf{R}^d$. As we now know that for such Λ , $\hat{\nu}(\Lambda)$ is finite, we can just mimic the proof of Lemma 3.2: Observe first that if m is finite and $\Lambda_m = \{a \in \Lambda : |a| \leq m\}$, then L^{Λ_m} can only become infinite by making infinitely many jumps in finite time, and since $\hat{\nu}(\Lambda)$ is finite, this only happens with probability 0 (compare the proof of Lemma 3.2). Hence L^{Λ_m} is a hyperfinite Lévy process. That L^Λ is also an hyperfinite Lévy process, now follows exactly as in the proof of Lemma 3.2. ♠

Remark: It is easy to see that the result above does not hold for internal sets Λ in general — if we just remove one leg of Anderson’s random walk, the process will now longer stay finite!

We have now reached our characterization of hyperfinite Lévy processes.

Theorem 4.3 (Characterization of hyperfinite Lévy processes) *A hyperfinite random walk L is a hyperfinite Lévy process if and only if the following three conditions are satisfied:*

- (i) $\frac{1}{\Delta t} \sum_{|a| \leq k} ap_a$ is finite for all finite and noninfinitesimal $k \in {}^*\mathbf{R}$.
- (ii) $\frac{1}{\Delta t} \sum_{|a| \leq k} |a|^2 p_a$ is finite for all finite $k \in {}^*\mathbf{R}$.
- (iii) $\lim_{k \rightarrow \infty} {}^\circ q_k = 0$ in the sense that for every $\epsilon \in \mathbf{R}_+$, there is an $N \in \mathbf{N}$ such that $q_k < \epsilon$ when $k \geq N$.

Proof: Assume first that L is a hyperfinite Lévy process. Since condition (iii) is just the conclusion of Lemma 3.1, we may concentrate on (i) and (ii). Assume that k is finite and noninfinitesimal. According to the corollary above, $L^{k \leq}$ is a hyperfinite Lévy process with limited increments. This means that

$$\frac{1}{\Delta t} \sum_{|a| \leq k} ap_a = \mu_{L^{k \leq}}$$

and

$$\frac{1}{\Delta t} \sum_{|a| \leq k} |a|^2 p_a = \sigma_{L^{k \leq}}^2$$

are finite by Corollary 2.4. It only remains to prove (ii) for infinitesimal k , but this is trivial since $\frac{1}{\Delta t} \sum_{|a| \leq k} |a|^2 p_a$ is increasing with k .

For the converse, we assume that L is a hyperfinite random walk satisfying (i)-(iii). For any finite, noninfinitesimal k , conditions (i) and (ii) say that $\mu_{L^{k \leq}}$ and $\sigma_{L^{k \leq}}$ are finite and hence $L^{k \leq}$ is a hyperfinite Lévy process by Corollary 2.4. To prove that also L is a hyperfinite Lévy process, we use condition (iii) and argue exactly as in the last part of the proof of Lemma 3.2. ♠

Observe that condition (i) in the theorem is not required to hold for infinitesimal k . The following example shows that there are, in fact, hyperfinite Lévy

processes such that (i) fails for all sufficiently large infinitesimals k . As we shall see later, this will require us to be rather careful in dealing with the “diffusion part” of a hyperfinite Lévy process.

Example 3 Let $A = \{-\epsilon, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{N}\}$ where N is an element in ${}^*\mathbf{N} \setminus \mathbf{N}$ such that $N\Delta t \approx 0$, and where ϵ is a positive infinitesimal that I shall soon specify. We let $p_{\frac{1}{n}} = \Delta t$ and $p_{-\epsilon} = 1 - N\Delta t$. We want to choose ϵ such that L becomes a martingale. For this we need

$$\epsilon(1 - N\Delta t) = \Delta t \sum_{n=1}^N \frac{1}{n}$$

which yields

$$\epsilon = \frac{\Delta t \sum_{n=1}^N \frac{1}{n}}{1 - N\Delta t}$$

Note that ϵ is infinitesimal. Since

$$\sigma_L^2 = \frac{1}{\Delta t} \sum_{n=1}^N \frac{1}{n^2} \Delta t + \frac{1}{\Delta t} \left(\frac{\Delta t \sum_{n=1}^N \frac{1}{n}}{1 - N\Delta t} \right)^2 (1 - N\Delta t)$$

is finite (note that by the choice of N , the second term in the middle expression is infinitesimal), and $\mu_L = 0$ (since L is a martingale), L is a hyperfinite Lévy process by Corollary 2.4. Note, however, that for any infinitesimal k larger than ϵ ,

$$\frac{1}{\Delta t} \sum_{|a|>k} ap_a = \sum \left\{ \frac{1}{n} \mid n \leq N \text{ and } n < \frac{1}{k} \right\}$$

is infinite. Since $\sum_{a \in A} ap_a = 0$, this means that

$$\frac{1}{\Delta t} \sum_{|a| \leq k} ap_a = -\frac{1}{\Delta t} \sum_{|a| > k} ap_a$$

is (negative) infinite for all infinitesimal k larger than ϵ . This also means that for such values of k , $L^{\leq k}$ and $L^{k>}$ can not be hyperfinite Lévy processes. ♠

5 Decomposing a hyperfinite Lévy process

For many purposes it would be convenient to write L as a sum

$$L_t = I_t + S_t + B_t$$

where I is the sum of the infinitesimal increments of L , S the sum of the “small”, but noninfinitesimal increments of L , and B the sum of the “big” increments of L . To split the “big” and the “small” increments is often useful

for integrability purposes, but the dividing line is quite arbitrary, and we shall just put $B = L^{>k}$ for some (arbitrary) finite and noninfinitesimal k . We are then left with the problem of splitting a hyperfinite Lévy process with limited increments (namely $L^{\leq k}$) into two parts I and S . This is a much subtler problem for two reasons. The first is that since it is impossible to distinguish between infinitesimals and noninfinitesimals in an internal way, we have to allow some infinitesimal contributions in S (the alternative — to allow noninfinitesimal contributions in I — seems less attractive as noninfinitesimal contributions can not be neglected). The hope is that we can do this in such a way that the infinitesimal contributions to S are insignificant. The second problem is that although the drift coefficient $\mu_{L^{\leq k}}$ is finite, Example 3 shows that the drift coefficient of any “infinitesimal part” of $L^{\leq k}$ may be infinite. This means that we have to be very careful in handling the drift terms.

A few definitions before we begin: A *hyperfinite Lévy martingale* is just a hyperfinite Lévy process which is also an internal martingale with respect to the natural (internal) filtration $\{\mathcal{F}_t\}$. A hyperfinite Lévy process has *infinitesimal increments* if $|a| \approx 0$ for all $a \in A$. Finally, a hyperfinite Lévy process L is called a *hyperfinite jump process* if for any finite $t \in T$ and any $\epsilon \in \mathbf{R}_+$, there is a $\delta \in \mathbf{R}_+$ such that

$$E[\max_{s \leq t} |L^{\leq \delta}(s)|^2] < \epsilon$$

The idea is that although a hyperfinite jump process may have infinitesimal increments, their total contribution is insignificant. A *hyperfinite jump martingale* is, of course, a hyperfinite jump process that happens to be an internal martingale. The result we are aiming for in this section is:

Theorem 5.1 (Decomposing hyperfinite Lévy processes) *Assume that L is a hyperfinite Lévy process with limited increments. Then*

$$L(t) = \mu_L t + I(t) + S(t)$$

where $\mu_L \in {}^*\mathbf{R}^d$ is finite, I is a hyperfinite Lévy martingale with infinitesimal increments, and S is a hyperfinite jump martingale.

To approach this theorem, let L be a hyperfinite Lévy process with limited increments and decomposition

$$L(t) = \mu_L t + M_L(t)$$

For any positive ϵ (finite or infinitesimal), we may decompose $L^{\leq \epsilon}$ and $L^{>\epsilon}$ in a drift term and a martingale term in the same way:

$$L^{\leq \epsilon}(t) = \mu_{\leq \epsilon} t + I_\epsilon(t)$$

$$L^{>\epsilon}(t) = \mu_{>\epsilon} t + S_\epsilon(t)$$

(simplifying the notation to avoid too many complicated indices). Since $L(t) = L^{\leq \epsilon}(t) + L^{>\epsilon}(t)$, we get

$$L(t) = (\mu_{\leq \epsilon} + \mu_{>\epsilon})t + I_\epsilon(t) + S_\epsilon(t) \tag{1}$$

showing that $\mu_L = \mu_{\leq \epsilon} + \mu_{> \epsilon}$ and $M_L(t) = I_\epsilon(t) + S_\epsilon(t)$. Note that μ_L is finite (since L is a hyperfinite Lévy process with limited increments), but that $\mu_{\leq \epsilon}$ and $\mu_{> \epsilon}$ may be infinite when ϵ is infinitesimal. The idea is to obtain the decomposition in Theorem 5.1 by choosing a sufficiently large, infinitesimal ϵ in formula (1).

To find such an infinitesimal, we first note that for *noninfinitesimal* ϵ , the expression

$$\sigma_\epsilon^2 := \frac{1}{\Delta t} \sum_{a \leq \epsilon} a^2 p_a$$

is finite and decreases as ϵ decreases. Let

$$\beta = \inf\{\sigma_\epsilon^2 : 0 < \epsilon\}$$

The set $\{\epsilon \in {}^*\mathbf{R}_+ : \sigma_\epsilon^2 > \beta - \epsilon\}$ is internal and contains all noninfinitesimal numbers — hence it must also contain all sufficiently large infinitesimals. Such an infinitesimal is called a *splitting infinitesimal*. We now choose a splitting infinitesimal η so large that $|\mu_{> \eta}|^2 \Delta t$ and $|\mu_{\leq \eta}|^2 \Delta t$ are infinitesimal and that $|\mu_{> \eta}| \Delta t$ is less than η (since $|\mu_{> \epsilon}|$ and $|\mu_{\leq \epsilon}|$ are finite for all noninfinitesimal ϵ , this is clearly possible). Note that since η is a splitting infinitesimal,

$$\inf\left\{\frac{1}{\Delta t} \sum_{\eta \leq a \leq \epsilon} a^2 p_a : 0 < \epsilon\right\} = 0 \quad (2)$$

We now define our decomposition by

$$I = I_\eta \quad \text{and} \quad S = S_\eta$$

and are ready to show that these two processes satisfy the requirements of the theorem:

Proof of Theorem 5.1: We first observe that the increments of I are either of the form $a - \mu_{\leq \eta} \Delta t$ (if $|a| \leq \eta$) or of the form $-\mu_{\leq \eta} \Delta t$ (if $|a| > \eta$) and hence infinitesimal in both cases. To see that I is a hyperfinite Lévy process, we first observe that

$$\sigma_I^2 = \frac{1}{\Delta t} \left(\sum_{a \leq \eta} |a - \mu_{\leq \eta} \Delta t|^2 p_a + \sum_{a > \eta} |-\mu_{\leq \eta} \Delta t|^2 p_a \right)$$

is finite by choice of η . Since $\mu_I = 0$, Corollary 2.4 tells us that I is a hyperfinite Lévy process. Note that this also means that $S_t = L_t - I_t - \mu_L t$ is a hyperfinite Lévy process.

It only remains to show that S is a hyperfinite jump martingale. Since S is a martingale by construction, we only need to show that for any finite $t \in T$ and any $\epsilon \in \mathbf{R}_+$, there is a $\delta \in \mathbf{R}_+$ such that

$$E[\max_{s \leq t} |S^{\leq \delta}(s)|^2] < \epsilon$$

Since Doob's inequality (see, e.g., [1], [24]) tells us that

$$E[\max_{s \leq t} |S^{\leq \delta}(s)|^2] \leq 4E[|S^{\leq \delta}(t)|^2],$$

it clearly suffices to show that we can get $E[|S^{\leq \delta}(t)|^2]$ less than any positive, real number by choosing $\delta \in \mathbf{R}_+$ small enough. If we let B and $\{p_b\}_{b \in B}$ be the increments and the transition probabilities of S , respectively, we see from Lemma 1.2 that

$$E[|S^{\leq \delta}(t)|^2] = \hat{\sigma}_\delta^2 t$$

where

$$\hat{\sigma}_\delta^2 := \frac{1}{\Delta t} \sum_{b \leq \delta} b^2 p_b$$

Observe that every $b \in B$ corresponds in a natural way to an $a \in A$. If $|a| \leq \eta$, then $b = -\mu_{>\eta} \Delta t$, and if $|a| > \eta$, then $b = a - \mu_{>\eta} \Delta t$. Thus

$$\hat{\sigma}_\delta^2 = \frac{1}{\Delta t} \sum_{|b| \leq \delta} b^2 p_b = \frac{1}{\Delta t} \sum_{|a| \leq \eta} |-\mu_{>\eta} \Delta t|^2 p_a + \frac{1}{\Delta t} \sum_{|a| > \eta \text{ and } |b| \leq \delta} |a - \mu_{>\eta} \Delta t|^2 p_a$$

The first term on the right is infinitesimal (recall that we have chosen η such that $|\mu_{>\eta}|^2 \Delta t$ is infinitesimal). Our task is to show that we can get the second term less than any positive real number. Observe that since $|b| \leq \delta$, we must have $|a| \leq \delta + |\mu_{>\eta}| \Delta t < 2\delta$, and that since $|a| > \eta > |\mu_{>\eta}| \Delta t$ (recall that we have chosen η such that $|\mu_{>\eta}| \Delta t$ is less than η), we also have $|a - \mu_{>\eta} \Delta t| \leq |a| + |\mu_{>\eta}| \Delta t < 2|a|$. Thus

$$\frac{1}{\Delta t} \sum_{|a| > \eta \text{ and } |b| \leq \delta} |a - \mu_{>\eta} \Delta t|^2 p_a \leq \frac{1}{\Delta t} \sum_{\eta < |a| \leq 2\delta} (2|a|)^2 p_a$$

By formula (2), we can get this expression as small as we want, and the proof of the theorem is complete. \spadesuit

The idea behind Theorem 5.1 is that I will be the continuous part and S the “pure jump” part of the process L . It is not entirely obvious that I is continuous, but this follows from the next result:

Proposition 5.2 *A hyperfinite Lévy process with infinitesimal increments is S-continuous.*

Proof: It clearly suffices to show that all hyperfinite Lévy martingales M with infinitesimal increments are S-continuous. Since $E(|\Delta M(t)|^2 | \mathcal{F}_t) = \sigma_M^2 \Delta t$, we see that the bracket process

$$\langle M \rangle(t) := \sum_{s < t} E[|\Delta M(s)|^2 | \mathcal{F}_t] = \sigma_M^2 t$$

is S-continuous. Since M has infinitesimal increments, Theorem 8.5c in Hoover and Perkins [19] (reproduced as Theorem 8.8 in [24]) tells us that M is S-continuous. ♠

Remark If L is a one-dimensional process, the proposition above in combination with [24, Theorem 11.3] tells us that the standard part \mathcal{M} of M is of the form $\mathcal{M}_t = {}^\circ\sigma_M b_t$ for a Brownian motion b . For higher dimensional processes we similarly have $\mathcal{M}_t = D_M b_t$ for a matrix D_M . This is quite easy to prove (e.g. by computing Fourier transforms), but as it will follow from the nonstandard Lévy-Khintchine formula that we prove in Section 8, we do not spend time on it here.

6 Standard parts

So far we have been looking at our processes from a strictly nonstandard perspective. Time has come to relate our theory to the standard theory of Lévy processes. We first want to turn our hyperfinite Lévy process L into a standard process by taking standard parts. Since L in general will have noninfinitesimal jumps, and we want the standard part to be right continuous with left limits, we have to be a little careful with our definitions. We follow the treatment in [1] and [24].

Definition 6.1 Assume that $F : T \rightarrow {}^*\mathbf{R}^d$ is an internal function. Let $r \in [0, \infty)$ and $b \in \mathbf{R}^d$. We say that b is the S-right limit of F at r if for every $\epsilon \in \mathbf{R}_+$ there is a $\delta \in \mathbf{R}_+$ such that if $t \in T$ satisfies $r < {}^\circ t < r + \delta$, then $|F(t) - b| < \epsilon$. We write

$$S\text{-}\lim_{s \downarrow r} F(s) = b$$

The S-left limit, $S\text{-}\lim_{s \uparrow r} F(s)$, is defined similarly.

If an internal function has S-right and S-left limits at all $r \in [0, \infty)$, we say that it has *one-sided limits*. An internal process $X : \Omega \times T \rightarrow {}^*\mathbf{R}^d$ has one-sided limits if P_L -almost all the paths $t \rightarrow X(\omega, t)$ have one-sided limits.

Definition 6.2 If $F : T \rightarrow {}^*\mathbf{R}^d$ has one-sided limits, its (right) standard part is the function ${}^\circ F : [0, \infty) \rightarrow \mathbf{R}^d$ defined by

$${}^\circ F(r) = S\text{-}\lim_{s \downarrow r} F(s)$$

Remark It is easy to check that ${}^\circ F$ is a right continuous function with left limits. In fact, ${}^\circ F$ is the standard part of F in the Skorohod topology on the space of right continuous functions with left limits (see Hoover and Perkins [19] and Stroyan and Bayod [36]). In our main reference [1], the right standard part is denoted by ${}^\circ F^+$.

Proposition 6.3 *A hyperfinite Lévy process L has one-sided limits.*

Proof: If L has limited increments, this follows immediately from the decomposition $L(\omega, t) = \mu_L t + M_L(\omega, t)$ and the fact that square integrable, internal martingales have one-sided limits (see, e.g., [1, Proposition 4.2.10] or [24, Theorem 7.6]). The general case follows by approximating by processes with limited increments (recall Proposition 3.3). ♠

We shall write \mathcal{L} for the standard part of L , i.e. $\mathcal{L} = {}^\circ L$. Our goal in this section is to show that \mathcal{L} is a standard Lévy process. To do this, we need the following lemma (which basically says that \mathcal{L} is continuous in probability).

Lemma 6.4 *For each $\epsilon \in \mathbf{R}_+$ there is a $\delta \in \mathbf{R}_+$ such that whenever $s, t \in T$, $|s - t| < \delta$, then*

$$P[|L(t) - L(s)| \geq \epsilon] < \epsilon$$

In particular, if $r \in [0, \infty)$ and $s \in T$ is infinitely close to r , then ${}^\circ L(t) = \mathcal{L}(r)$ P_L -almost everywhere.

Proof: By Proposition 3.3 it clearly suffices to prove this for hyperfinite Lévy processes with limited increments. For $t > s$ we have by Lemma 1.2

$$E[|L_t - L_s|^2] = \sigma_L^2(t - s) + |\mu_L|^2(t - s)(t - s - \Delta t)$$

The first statement now follows from Chebyshev's inequality:

$$P[|L_t - L_s| \geq \epsilon] \leq \frac{E[|L_t - L_s|^2]}{\epsilon^2} = \frac{\sigma_L^2(t - s) + |\mu_L|^2(t - s)(t - s - \Delta t)}{\epsilon^2}$$

To prove the second statement, observe that if $\{t_n\}_{n \in \mathbf{N}}$ is a sequence of elements in T such that ${}^\circ t_n \downarrow r$, the sequence $\{{}^\circ L_{t_n}\}_{n \in \mathbf{N}}$ converges to ${}^\circ L_t$ in P_L -probability by what we have already proved. Since the same sequence converges to \mathcal{L}_r almost surely (by the definition of \mathcal{L} and the existence of S-right limits of L), we must have ${}^\circ L_t = \mathcal{L}_r$ P_L -almost everywhere. ♠

We shall use the following definition of (standard) Lévy processes:

Definition 6.5 *A stochastic process $X : \Omega \times [0, \infty) \rightarrow \mathbf{R}^d$ is called a Lévy process if:*

- (i) *X has independent increments, i.e. if $0 < t_0 < t_1 < t_2 \dots < t_n$, then the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.*
- (ii) *$X(0) = 0$ a.s.*
- (iii) *X is time homogeneous, i.e. the distribution of $X_{t+s} - X_t$ does not depend on t .*
- (iv) *Almost all the paths of X are right continuous with left limits.*

We have now reached our goal in this section:

Theorem 6.6 *\mathcal{L} is a Lévy process.*

Proof: (i) This follows from the fact that L has $*$ -independent increments and that (by the lemma) non-overlapping increments of \mathcal{L} can be represented by non-overlapping increments of L .
(ii) Since $L(0) = 0$, this follows from the lemma.
(iii) This is obvious from the construction.
(iv) As already observed, this is a consequence of Proposition 6.3. ♠

Although we now know that our hyperfinite Lévy processes give rise to standard Lévy processes, we still don't have a good understanding of the relationship between the two classes of processes — e.g., if we want \mathcal{L} to have certain properties, how should we choose the increments A and the transition probabilities $\{p_a\}_{a \in A}$ of L in order to achieve this? To answer this question, we must take a closer look at the diffusion part and the jump part of L .

7 Lévy measures and covariance matrices

Our main bridge connecting the standard and the nonstandard theory will be the hyperfinite Lévy-Khintchine formula which we shall prove in the next section. This section contains some preliminary material that will be useful in stating and proving this formula. First we take a look at the Lévy measure.

Let L be a hyperfinite Lévy process. In section 4 we introduced an internal measure $\hat{\nu}$ on ${}^*\mathbf{R}^d$ by

$$\hat{\nu}(B) = \frac{1}{\Delta t} \sum_{a \in B} p_a$$

and proved that $\hat{\nu}(B)$ is finite as long as B does not contain any infinitesimal elements. As usual we let $\hat{\nu}_L$ be the Loeb measure of $\hat{\nu}$. By a well-known procedure (see, e.g., chapter 3 of [1]), the measure $\hat{\nu}_L$ on ${}^*\mathbf{R}^d$ can be “pushed down” to a completed Borel measure ν on \mathbf{R}^d . This is done simply by letting

$$\nu(C) = \hat{\nu}_L(st^{-1}(C))$$

whenever the expression on the right makes sense. The measure ν will give infinite mass to the origin, and since this is rather inconvenient, we shall now modify ν so that it does not charge the origin. For ν -measurable sets $C \subset \mathbf{R}^d$, let

$$C_\epsilon = \{x \in C : |x| \geq \epsilon\}$$

and define the *Lévy measure* $\nu_{\mathcal{L}}$ by

$$\nu_{\mathcal{L}}(C) = \lim_{\epsilon \downarrow 0} \nu(C_\epsilon)$$

It is easy to check that $\nu_{\mathcal{L}}$ is a completed Borel measure on \mathbf{R}^d .

We shall refer to $\nu_{\mathcal{L}}$ as the *Lévy measure* of \mathcal{L} . It will become clear in the next section that it really is the Lévy measure of \mathcal{L} in the ordinary sense. The first step is the following result which readers familiar with the (standard) theory of Lévy processes will recognize as the standard characterization of a Lévy measure:

Proposition 7.1 *The Lévy measure $\nu_{\mathcal{L}}$ has the following properties:*

$$(i) \nu_{\mathcal{L}}(\{0\}) = 0$$

$$(ii) \int_{\{x: |x| \leq 1\}} |x|^2 d\nu_{\mathcal{L}}(x) < \infty$$

$$(iii) \nu_{\mathcal{L}}(\{x \in \mathbf{R}^d : |x| \geq 1\}) < \infty$$

Note that (ii) and (iii) can be combined as

$$(ii+iii) \int_{\mathbf{R}^d} (|x|^2 \wedge 1) d\nu_{\mathcal{L}}(x) < \infty$$

Proof: (i) Follows directly from the definition of $\nu_{\mathcal{L}}$.

(ii) By basic Loeb measure theory

$$\int_{[\epsilon \leq |x| \leq 1]} |x|^2 d\nu_{\mathcal{L}}(x) \leq \int_{[\epsilon/2 \leq x \leq 1+\epsilon]} |a|^2 d\hat{\nu}(a) \leq \left(\frac{1}{\Delta t} \sum_{|a| \leq 2} |a|^2 p_a \right) < \infty$$

for all small $\epsilon \in \mathbf{R}_+$. If we let $\epsilon \downarrow 0$, the expression on the left tends to $\int_{\{x: |x| \leq 1\}} |x|^2 d\nu_{\mathcal{L}}(x)$ by the Monotone Convergence Theorem.

(iii) By the construction of $\nu_{\mathcal{L}}$ we have

$$\nu_{\mathcal{L}}(\{x \in \mathbf{R}^d : |x| \geq 1\}) \leq \hat{\nu}(\{x \in {}^*\mathbf{R}^d : |x| \geq \frac{1}{2}\}) < \infty \quad \spadesuit$$

In section 5, we showed that it is possible to find a splitting infinitesimal $\eta > 0$ such that

$$\inf \left\{ \frac{1}{\Delta t} \sum_{\eta \leq a \leq \epsilon} a^2 p_a : 0 < \epsilon < \eta \right\} = 0$$

In terms of $\hat{\nu}$, this can be rewritten

$$\inf \left\{ \int_{B_{[\eta, \epsilon]}} a^2 d\hat{\nu}(a) : 0 < \epsilon < \eta \right\} = 0$$

where

$$B_{[\eta, \epsilon]} = \{x \in {}^*\mathbf{R}^d : \eta \leq x \leq \epsilon\}$$

(since we think of $\hat{\nu}$ as an internal measure on ${}^*\mathbf{R}^d$, we write $\int_C f(a) d\hat{\nu}(a)$ for $\sum_{a \in C} f(a) \hat{\nu}(\{a\})$).

The following result will be helpful in the next section.

Proposition 7.2 *Assume that the internal function $F : {}^*\mathbf{R}^d \rightarrow {}^*\mathbf{R}$ is S -continuous at all finite and noninfinitesimal $a \in {}^*\mathbf{R}^d$, and that there is a $C \in \mathbf{R}_+$ such that $|F(a)| \leq C(|a|^2 \wedge 1)$ for all $a \in {}^*\mathbf{R}^d$. Then*

$$\int_{\mathbf{R}^d} {}^\circ F(x) d\nu_{\mathcal{L}}(x) = \int_{|a| > \eta} F(a) d\hat{\nu}(a) < \infty$$

where η is a splitting infinitesimal (as above).

Proof: It clearly suffices to prove the result for nonnegative functions F . Observe that for all $n \in \mathbf{N}$, we have

$$\begin{aligned} \circ \int_{\{\frac{2}{n} \leq |a| \leq n - \frac{1}{n}\}} F(a) d\hat{\nu}(a) &\leq \int_{\{\frac{1}{n} \leq |x| \leq n\}} \circ F(x) d\nu_{\mathcal{L}}(x) \leq \\ &\leq \circ \int_{\{\frac{1}{2n} \leq |a| \leq n + \frac{1}{n}\}} F(a) d\hat{\nu}(a) \end{aligned}$$

by construction of $\nu_{\mathcal{L}}$ and basic Loeb measure theory. When $n \rightarrow \infty$, the first and the last expression converge to $\circ \int_{|a| > \eta} F(a) d\hat{\nu}(a) < \infty$ (here we are using the definition of η and the bounds on the function F), while the one in the middle converges to $\int_{\mathbf{R}^d} \circ F(x) d\nu_{\mathcal{L}}(x)$ by the Monotone Convergence Theorem.

♠

The Lévy measure will help us to study the jump part of L . The continuous part is best described by a covariance matrix. We let x_i denote the i -th component of a vector $x \in {}^*\mathbf{R}^d$ (in particular, L_i denotes the i -th component of the ${}^*\mathbf{R}^d$ -valued process L):

Lemma 7.3 *For all hyperfinite random walks L :*

$$E[L_i(t)L_j(t)] = \frac{t}{\Delta t} \sum_{a \in A} a_i a_j p_a + \mu_i \mu_j t(t - \Delta t)$$

Proof: This is just as the proof of Lemma 1.2:

$$\begin{aligned} E[L_i(t)L_j(t)] &= E\left[\sum_{s < t} \Delta L_i(s) \sum_{r < t} \Delta L_j(r)\right] = \\ &= \sum_{s < t} E[\Delta L_i(s) \Delta L_j(s)] + \sum_{0 \leq s \neq r < t} E[\Delta L_i(s) \Delta L_j(r)] = \\ &= \frac{t}{\Delta t} \sum_{a \in A} a_i a_j p_a + \mu_i \mu_j t(t - \Delta t) \quad \spadesuit \end{aligned}$$

The $d \times d$ -matrix C^L with elements

$$C_{ij}^L = \frac{1}{\Delta t} \sum_{a \in A} a_i a_j p_a$$

is called the *infinitesimal covariance matrix* of L .

Lemma 7.4 *C^L is symmetric, nonnegative definite and*

$$\langle C^L x, x \rangle = \frac{1}{\Delta t} \sum_{a \in A} \langle a, x \rangle^2 p_a \leq \sigma_L^2 |x|^2$$

where $\langle \cdot, \cdot \rangle$ is the inner product in \mathbf{R}^d .

Proof: The matrix is obviously symmetric, and the nonnegative definiteness follows from the formula. Hence all we need is the following calculation:

$$\begin{aligned}\langle C^L x, x \rangle &= \sum_{i,j} C_{ij}^L x_i x_j = \frac{1}{\Delta t} \sum_{i,j} \sum_{a \in A} a_i a_j p_a x_i x_j = \\ &= \frac{1}{\Delta t} \sum_{a \in A} \sum_{i,j} a_i x_i a_j x_j p_a = \frac{1}{\Delta t} \sum_{a \in A} \langle a, x \rangle^2 p_a \leq \frac{1}{\Delta t} \sum_{a \in A} |a|^2 |x|^2 p_a = \sigma_L^2 |x|^2\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. ♠

8 The hyperfinite Lévy-Khintchine formula

The (standard) Lévy-Khintchine formula is just an expression for the Fourier-transform of an arbitrary Lévy process. Here we have a similar formula for hyperfinite Lévy processes:

Theorem 8.1 (Hyperfinite Lévy-Khintchine formula) *Assume that L is a hyperfinite Lévy process. Let $k \in {}^*\mathbf{R}_+$ be finite and noninfinitesimal, and let η be a splitting infinitesimal (i.e. $\inf\{\frac{1}{\Delta t} \sum_{\eta \leq a \leq \epsilon} a^2 p_a : 0 < \epsilon\} = 0$). Then for all finite $y \in {}^*\mathbf{R}^d$:*

$$\begin{aligned}E[e^{i\langle y, L_t \rangle}] &\approx \exp \left[it \langle y, \mu_k \rangle - \frac{t}{2} \langle C^\eta y, y \rangle \right. \\ &\quad \left. + t \int_{|a| > \eta} \left(e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle 1_{[|a| \leq k]} \right) d\hat{\nu}(a) \right]\end{aligned}$$

where C^η is the infinitesimal covariance matrix of the process $L^{\leq \eta}$ and where $\mu_k := \mu_{L^{\leq k}}$.

Proof: We first observe that

$$E \left[e^{i\langle y, L_t \rangle} \right] = \prod_{s < t} E \left[e^{i\langle y, \Delta L(s) \rangle} \right] = E \left[e^{i\langle y, \Delta L(0) \rangle} \right]^{\frac{t}{\Delta t}}$$

If we can prove that

$$E \left[e^{i\langle y, \Delta L(0) \rangle} \right] = 1 + R\Delta t + o(\Delta t)$$

where R is finite and $o(\Delta t)$ is infinitesimal compared to Δt , it follows by non-standard calculus that

$$E \left[e^{i\langle y, L(t) \rangle} \right] = E \left[e^{i\langle y, \Delta L(0) \rangle} \right]^{\frac{t}{\Delta t}} = (1 + R\Delta t + o(\Delta t))^{\frac{t}{\Delta t}} \approx e^{Rt}$$

Hence our task is to show that

$$\begin{aligned} E \left[e^{i\langle y, \Delta L(0) \rangle} \right] &= 1 + \Delta t \left[i\langle y, \mu_k \rangle - \frac{1}{2} \langle C^\eta y, y \rangle + \right. \\ &\quad \left. + \int_{|a| > \eta} \left(e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle 1_{[|a| \leq k]} \right) d\hat{\nu}(a) \right] + o(\Delta t) \end{aligned}$$

To get $E[e^{i\langle y, \Delta L(0) \rangle}]$ on the form “1 + something infinitesimal”, it is natural to write

$$E \left[e^{i\langle y, \Delta L(0) \rangle} \right] = 1 + E \left[e^{i\langle y, \Delta L(0) \rangle} - 1 \right]$$

This is not quite enough to give us the estimates we want, and it is tempting to subtract the first order term $i\langle y, \Delta L(0) \rangle$ of the exponential $e^{i\langle y, \Delta L(0) \rangle}$ inside the last expectation. Since the process L may fail to be integrable, this is rather dangerous, and we choose to work with the truncated process $L^{\leq k}$ instead. Indeed, since $L^{\leq k}(t) - \mu_k t$ is a martingale, we have

$$\begin{aligned} E \left[e^{i\langle y, \Delta L(0) \rangle} \right] &= 1 + E \left[e^{i\langle y, \Delta L(0) \rangle} - 1 - i\langle y, \Delta L(0)^{\leq k} - \mu_k \Delta t \rangle \right] = \\ &= 1 + i\langle y, \mu_k \rangle \Delta t + E \left[e^{i\langle y, \Delta L(0) \rangle} - 1 - i\langle y, \Delta L(0)^{\leq k} \rangle \right] = \\ &= 1 + i\langle y, \mu_k \rangle \Delta t + \sum_{a \in A} \left(e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle 1_{[|a| \leq k]} \right) p_a \end{aligned}$$

If we split the last sum at $a = \eta$ and use that $\hat{\nu}(a) = \frac{p_a}{\Delta t}$, the expression above can be rewritten as

$$\begin{aligned} E \left[e^{i\langle y, \Delta L(0) \rangle} \right] &= 1 + i\langle y, \mu_k \rangle \Delta t + \sum_{|a| \leq \eta} \left(e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle \right) p_a + \\ &\quad + \Delta t \int_{|a| > \eta} \left(e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle 1_{[|a| \leq k]} \right) d\hat{\nu}(a) \end{aligned}$$

The last term in this expression is exactly what we want, and if combine Taylor expansion and Lemma 7.4, we see that the penultimate term can be rewritten as

$$\begin{aligned} \sum_{|a| \leq \eta} \left(e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle \right) p_a &\approx - \sum_{|a| \leq \eta} \frac{1}{2} \langle y, a \rangle^2 p_a + o(\Delta t) = \\ &= -\frac{1}{2} \langle C^\eta y, y \rangle \Delta t + o(\Delta t) \end{aligned}$$

Hence

$$\begin{aligned} E \left[e^{i\langle y, \Delta L(0) \rangle} \right] &= 1 + \Delta t \left[i\langle y, \mu_k \rangle - \frac{1}{2} \langle C^\eta y, y \rangle + \right. \\ &\quad \left. + \int_{|a| > \eta} \left[e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle 1_{[|a| \leq k]} \right] d\hat{\nu}(a) \right] + o(\Delta t) \end{aligned}$$

Since $\langle C^\eta y, y \rangle$ is finite by Lemma 7.4 and the integral is finite by Proposition 7.2, the proof is complete. ♠

Note that the choice of the parameter k in the hyperfinite Lévy-Khintchine formula is of little importance — all that happens if we choose another k is that the change in the value of the integral will be compensated for by a different choice of μ_k (the measure $\nu_{\mathcal{L}}$ and the matrix C^η remain the same). For this reason it is usual to fix $k = 1$ in the standard Lévy-Khintchine formula.

It is also worthwhile to observe a few other consequences of the hyperfinite Lévy-Khintchine formula. We see, e.g., that although the processes I and S (recall the decomposition in Theorem 5.1) are *not* *-independent, their standard parts are independent (in the ordinary sense) since the Fourier transform of their sum equals the product of their Fourier transforms. We can also read from the formula that the standard part of I is a gaussian process with covariance matrix ${}^\circ C^\eta$.

In order to compare the hyperfinite and the standard versions of the Lévy-Khintchine formula, we first need to introduce some terminology: A *generating triplet* (γ, C, ν) consists of a $\gamma \in \mathbf{R}^d$, a real, symmetric, nonnegative definite $d \times d$ -matrix C and a completed Borel measure ν on \mathbf{R}^d satisfying the conditions in Proposition 7.1. The matrix C is called the *gaussian covariance matrix* and the measure ν is called the *Lévy measure*. Here is the standard version of the Lévy-Khintchine formula (see, e.g., [34]):

Theorem 8.2 (Standard Lévy-Khintchine formula) *Let \mathcal{L} be a Lévy process. There exists a generating triplet (γ, C, ν) such that for all t and all $y \in \mathbf{R}^d$*

$$\begin{aligned} E[e^{i\langle y, \mathcal{L}(t) \rangle}] &= \exp \left[it\langle y, \gamma \rangle - \frac{t}{2} \langle Cy, y \rangle \right. \\ &\quad \left. + t \int_{\mathbf{R}^d} \left(e^{i\langle y, a \rangle} - 1 - i\langle y, a \rangle 1_{[|a| \leq 1]} \right) d\nu(a) \right] \end{aligned}$$

Conversely, given a generating triplet (γ, C, ν) , we can find a Lévy process \mathcal{L} such that the formula above holds.

To compare the two versions of the Lévy-Khintchine formula, assume that L is a hyperfinite Lévy process with standard part \mathcal{L} . Comparing the formulas (and using Proposition 7.1), we see that $\nu_{\mathcal{L}}$ is the Lévy measure of \mathcal{L} in the standard sense. We also see that the gaussian covariance matrix C of \mathcal{L} is the standard part of the nonstandard matrix C^η . Hence we have proved:

Corollary 8.3 *The standard part of a hyperfinite Lévy process is a standard Lévy process with gaussian covariance matrix ${}^\circ C^\eta$ and Lévy measure $\nu_{\mathcal{L}}$.*

9 Representing standard Lévy processes

A natural question at this stage is whether all generating triplets can be produced by hyperfinite Lévy processes. Since two Lévy processes with the same

triplet have the same law, this is the same as asking whether all Lévy laws can be obtained from hyperfinite Lévy processes. Albeverio and Herzberg [2] have shown that all standard Lévy processes can be represented by hyperfinite random walks (in a slightly different sense than ours), and it is probably not difficult to show from their result that all Lévy laws can be obtained from hyperfinite Lévy processes (in our sense). However, it still seems interesting and useful to have an alternative proof based on generating triplets and the Lévy-Khintchine formula as this gives a good feeling for how we in practice can construct a hyperfinite representation with the properties that we want. Note that since we have quite a lot of freedom in the construction below, we may, e.g., build a representation that lives on a lattice with infinitesimal spacing.

Theorem 9.1 *Given a generating triplet (γ, C, ν) , there is a hyperfinite Lévy process L with standard part corresponding to this triplet.*

Proof: Although the idea is quite simple, the proof is technically and notationally a little messy. We first observe that it suffices to construct a process corresponding to $(\hat{\gamma}, C, \nu)$ for *some* vector $\hat{\gamma}$ as we can easily adjust $\hat{\gamma}$ by adding a drift. The main part of the proof consists of two steps. In the first step we use the measure ν to construct a hyperfinite set $A_1 \subset {}^*\mathbf{R}^d$ and transition probabilities $\{p_a\}_{a \in A_1}$. The elements in A_1 will form the “jump part” of L and they will all be larger than a certain (splitting) infinitesimal η which is infinite compared to $\sqrt{\Delta t}$. In the second part of the proof we use the matrix C to construct a finite set A_2 of elements in ${}^*\mathbf{R}^d$ with corresponding transition probabilities $\{p_a\}_{a \in A_2}$. The elements in A_2 will form the “diffusion part” of L , and they will all be of order of magnitude $\sqrt{\Delta t}$ (and hence smaller than the splitting infinitesimal η).

To begin step one, let ${}^*\nu$ be the nonstandard version of the given measure ν . For each $N \in {}^*\mathbf{N}$, let $B_N = \{x \in {}^*\mathbf{R}^d : \frac{1}{N} \leq |x| \leq N\}$. Since ${}^*\nu(B_N)$ is finite for all finite N , we see that ${}^*\nu(B_N)$ is infinitesimal compared to $\frac{1}{\Delta t}$ for all sufficiently small $N \in {}^*\mathbf{N} \setminus \mathbf{N}$. By a similar argument, we see that for all sufficiently small $N \in {}^*\mathbf{N} \setminus \mathbf{N}$, we have $|\int_{[\frac{1}{N} \leq |x| \leq 1]} |x| d{}^*\nu(x)|$ infinitesimal compared to $\frac{1}{\sqrt{\Delta t}}$. We now choose $N \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that both these conditions are satisfied, i.e.

$${}^*\nu(B_N) \text{ is infinitesimal compared to } \frac{1}{\Delta t} \quad (3)$$

and

$$\int_{[\frac{1}{N} \leq |x| \leq 1]} |x| d{}^*\nu(x) \text{ is infinitesimal compared to } \frac{1}{\sqrt{\Delta t}} \quad (4)$$

We also assume that N is so small that

$$\frac{1}{N} \text{ is infinitely large compared to } \sqrt{\Delta t} \quad (5)$$

and define our splitting infinitesimal by $\eta = \frac{1}{N}$.

We now discretize the set B_N by partitioning it into a hyperfinite family of sets with infinitesimal diameter and choosing one point a from each partition

class (this can be done, e.g., by using a lattice with infinitesimal spacing). Let A_1 be the (hyperfinite) set of all chosen points, and let $\hat{\nu}$ be the internal measure on A_1 defined by $\hat{\nu}(a) = {}^*\nu([a])$, where $[a]$ is the partition class of a . Observe that by (3), $\hat{\nu}(A_1)$ is infinitesimal compared to $1/\Delta t$. Hence if we define the transition probabilities p_a for $a \in A_1$ by

$$p_a = \hat{\nu}(a)\Delta t$$

we see that $\sum_{a \in A_1} p_a$ is infinitesimal. The remaining probability $q = 1 - \sum_{a \in A_1} p_a \approx 1$ will be used to construct the diffusion part of L .

Before we turn to the diffusion part, there are four observations we should make. The first is that

$$\frac{1}{\Delta t} \sum_{a \in A_1, |a| \leq k} |a|^2 p_a < \infty \quad (6)$$

for all finite k . This follows immediately from the construction and the condition $\int_{\mathbf{R}^d} (|x|^2 \wedge 1) d\nu < \infty$ on the Lévy measure ν . By basically the same argument, we also get the second observation:

$$\inf\left\{\frac{1}{\Delta t} \sum_{a \in A_1, |a| \leq \epsilon} |a|^2 p_a : 0 < \epsilon < \infty\right\} = 0 \quad (7)$$

If we recall the notation $q_k = \sum_{|a| > k} p_a$, the third observation is that

$$\lim_{k \rightarrow \infty} q_k = 0 \quad (8)$$

(where $k \in \mathbf{N}$ and the limit is interpreted in the standard sense). This follows from the fact that $\lim_{k \rightarrow \infty} \nu(\{x \in \mathbf{R}^d : |x| \geq k\}) = 0$. The fourth and final observation is that

$$\text{the norm of } \delta := \sum_{a \in A_1} a p_a \text{ is infinitesimal compared to } \frac{1}{\sqrt{\Delta t}} \quad (9)$$

which follows from condition (4) above (at least if we have chosen a discretization that is not too wild — it suffices, e.g., to require that if $b \in [a]$, then $|b| \geq |a|/2$).

Note that these four observations correspond to requirements we are already familiar with. The second observation (7) tells us that we can use η as a splitting infinitesimal, while the other three observations correspond to the three conditions characterizing hyperfinite Lévy processes in Theorem 4.3. Note that the last observation (9) is not quite what we need to get condition (i) in Theorem 4.3 — we would like to have the sum finite and not only less than the (infinite) quantity $\frac{1}{\sqrt{\Delta t}}$. In view of Example 3 (see section 4), this discrepancy is only to be expected, but we shall have to take it into account when we now turn to the construction of the diffusion part of L .

The set A_2 describing the diffusion part of L will consist of $2d$ elements (as usual d is the dimension of the Euclidean space we are working in), and all

these elements will have the same transition probability $p_a = \frac{q}{2d}$ (recall that $q = 1 - \sum_{a \in A_1} p_a \approx 1$ is the probability that has *not* been used for the jump part). When we choose the elements in A_2 , there are two things we have to take into account. The first is that the infinitesimal covariance matrix C^η should be infinitely close to the given matrix C , and the second is that we need to correct the (possibly) infinite drift δ introduced by the jump part (see (9)).

To construct A_2 , we first choose a $d \times d$ -matrix D such that $C = DD^T$. If $\{e_1, e_2, \dots, e_d\}$ is the standard basis in \mathbf{R}^d , we let A_2 consist of the $2d$ points

$$a^{\pm k} = \pm D e_k \sqrt{\frac{d\Delta t}{q}} - \delta \frac{\Delta t}{q}$$

where $k = 1, 2, \dots, d$, and where δ is as defined in (9). Observe that since $|\delta|$ is infinitesimal compared to $1/\sqrt{\Delta t}$, the last term in this expression is infinitesimal compared to the first one. Observe also that

$$\frac{1}{\Delta t} \sum_{|a| \in A_2} a p_a = -\delta \quad (10)$$

If we now let $A = A_1 \cup A_2$, we have formally defined a hyperfinite random walk with increments A and transition probabilities p_a . It remains to check that L is a hyperfinite Lévy process with Lévy measure ν and with covariance matrix C^η infinitely close to C (note that by (7), η can be used as a splitting infinitesimal in the hyperfinite Lévy-Khintchine formula).

We start with the covariance matrix. Observe that the i -th component of the vector $a^{\pm k}$ is given by

$$a_i^{\pm k} = \pm D_{i,k} \sqrt{\frac{d\Delta t}{q}} - \delta_i \frac{\Delta t}{q}$$

and using this to compute the covariance matrix, we get

$$\begin{aligned} C_{i,j}^\eta &= \frac{1}{\Delta t} \sum_{a \in A_2} a_i a_j p_a = \\ &= \frac{1}{\Delta t} \left[\sum_{k=1}^d \left(D_{i,k} \sqrt{\frac{d\Delta t}{q}} - \delta_i \frac{\Delta t}{q} \right) \left(D_{j,k} \sqrt{\frac{d\Delta t}{q}} - \delta_j \frac{\Delta t}{q} \right) \frac{q}{2d} + \right. \\ &\quad \left. + \sum_{k=1}^d \left(-D_{i,k} \sqrt{\frac{d\Delta t}{q}} - \delta_i \frac{\Delta t}{q} \right) \left(-D_{j,k} \sqrt{\frac{d\Delta t}{q}} - \delta_j \frac{\Delta t}{q} \right) \frac{q}{2d} \right] = \\ &= C_{i,j} + \delta_i \delta_j \frac{\Delta t}{q} \approx C_{i,j} \end{aligned}$$

where the last step uses that $|\delta|$ is infinitesimal compared to $1/\sqrt{\Delta t}$.

We next check that L is a hyperfinite Lévy process, i.e. that it satisfies the three conditions in Theorem 4.3. Condition (iii) is obviously satisfied by (8) above. Condition (ii) requires us to check that

$$\frac{1}{\Delta t} \sum_{|a| \leq k} |a|^2 p_a = \frac{1}{\Delta t} \sum_{a \in A_1, |a| \leq k} |a|^2 p_a + \frac{1}{\Delta t} \sum_{|a| \in A_2} |a|^2 p_a$$

is finite for all finite and (may we assume) noninfinitesimal k . But this is easy as the first sum is finite by (6), and the second is finite since $|a|$ has order of magnitude $\sqrt{\Delta t}$ for all $a \in A_2$. Condition (i) is the most sensitive. We first observe that

$$\frac{1}{\Delta t} \sum_{a \leq 1} a p_a = \frac{1}{\Delta t} \sum_{a \in A_1, a \leq 1} a p_a + \frac{1}{\Delta t} \sum_{a \in A_2} a p_a = \delta - \delta = 0$$

where we have used (9) and (10). To replace the cut-off 1 in the sum by an arbitrary finite and noninfinitesimal k as in condition (i), we just observe that

$$\frac{1}{\Delta t} \sum_{k_1 < |a| \leq k_2} a p_a = \int_{(k_1, k_2]} a d\hat{\nu}(a)$$

is finite for all finite and noninfinitesimal k_1, k_2 . Hence condition (i) is also satisfied, and L is a hyperfinite Lévy process.

It only remains to check that the Lévy measure $\nu_{\mathcal{L}}$ induced by L is the same as the original measure ν . But this follows from the construction of $\hat{\nu}$ using traditional Loeb measure techniques. ♠

As already mentioned, the theorem above tells us that as long as we classify processes by their law, all standard Lévy processes can be obtained as standard parts of hyperfinite Lévy processes. There are, of course, many other (and stronger) ways in which a standard process can be represented by a nonstandard process. Using lifting theorems and ideas from the model theory of stochastic processes (see [16]), Albeverio and Herzberg [2] have made a detailed study of the ways in which a standard Lévy process can be represented by a hyperfinite random walk. We shall not pursue this theme here, but refer the interested reader to [2].

10 A glimpse of Malliavin calculus

We have now completed the basic theory of hyperfinite Lévy processes. To get a feeling of what these processes may be used for, we shall end the paper with a brief and informal look at Malliavin calculus.

In this section, it is convenient to work with a finite timeline and a strict path space interpretation of our hyperfinite Lévy processes. To keep the notation simple, we shall also assume that our processes are one dimensional. We let $\Delta t = \frac{1}{N}$ for an infinite $N \in {}^*\mathbb{N}$ and define

$$T = \{0, \Delta t, 2\Delta t, \dots, 1\}$$

Given a set $A \subset {}^*\mathbf{R}$ of increments and an associated set $\{p_a\}_{a \in A}$ of transition probabilities, we let

$$\Omega = \{\omega : T \rightarrow A \mid \omega \text{ is internal}\}$$

and define P to be the internal probability measure on Ω given by

$$P(\{\omega\}) = \prod_{s \in T} p_{\omega(s)}$$

We let \mathcal{G} be the set of all internal subsets of Ω . Finally, we put

$$L(\omega, t) = \sum_{s < t} \omega_s$$

For any internal subset $\Gamma \subset T$, we let \mathcal{G}_Γ be the ${}^*\sigma$ -algebra generated by $\{\Delta L(t)\}_{t \in \Gamma}$. To keep notation short, we shall write $\mathcal{G}_{< t}$ for $\mathcal{G}_{\{s : s < t\}}$ and $\mathcal{G}_{\neq t}$ for $\mathcal{G}_{T \setminus \{t\}}$. Note that $\mathcal{G}_{< t}$ is identical to the σ -algebra we have so far denoted by \mathcal{F}_t . Given a path $\omega \in \Omega$, a time $t \in T$, and an increment $a \in A$, we let ω_t^a be the path we obtain when we replace the t -th increment of ω by a , i.e.

$$\omega_t^a = (\omega(0), \omega(\Delta t), \dots, \omega(t - \Delta t), a, \omega(t + \Delta t), \dots, \omega(1))$$

Note that the conditional expectation of an internal random variable X with respect to $\mathcal{G}_{\neq t}$ can be expressed as

$$E[X | \mathcal{G}_{\neq t}](\omega) = \sum_{a \in A} X(\omega_t^a) p_a$$

If $X : \Omega \times T \rightarrow {}^*\mathbf{R}$ is an internal process, we define the (internal) *Itô integral* by

$$\int_0^t X \, dL = \sum_{s < t} X(s) \, \Delta L(s)$$

By the nonstandard theory of stochastic integration (see [1] or [24] for expositions), this integral is well-behaved when X is $\mathcal{G}_{< t}$ -adapted (i.e. X_t is $\mathcal{G}_{< t}$ -measurable for all $t \in T$), but for general processes X , dependence between $X(s)$ and $\Delta L(s)$ may cause severe problems (e.g., that the integral becomes infinite in infinitesimal time although the integrand is finite). The hyperfinite *Skorohod integral* $\int X \diamond dL$ attempts to solve this problem by averaging out the most critical dependencies. It is defined by

$$\int_0^t X \diamond dL = \sum_{s < t} E[X | \mathcal{G}_{\neq t}] \Delta L(s)$$

Note that

$$\int_0^t X \diamond dL(\omega) = \sum_{s < t} \sum_{a \in A} X(\omega_s^a) p_a \Delta L(\omega, s)$$

Note also that if X is $\mathcal{G}_{<t}$ -adapted, then $\int X dL = \int X \diamond dL$. The *Malliavin divergence* δX of X is just the value of $\int X \diamond dL$ at the endpoint, i.e

$$\delta X = \int_0^1 X \diamond dL$$

If $x : \Omega \rightarrow {}^*\mathbf{R}$ is an internal random variable, the *Malliavin gradient* of x is the stochastic process $Dx : \Omega \times T \rightarrow {}^*\mathbf{R}$ given by

$$Dx(\omega, t) = \frac{1}{\sigma_L^2 \Delta t} E[x \Delta L(t) | \mathcal{G}_{\neq t}](\omega) = \frac{1}{\sigma_L^2 \Delta t} \sum_{a \in A} x(\omega_t^a) a p_a \quad (11)$$

(at this stage it would not be unnatural to assume that σ_L is finite, but this is actually not necessary for what follows).

Remark The reader may find this definition rather unexpected at first glance, but there are several reasons for claiming that it is the “correct” definition of the Malliavin gradient in our setting. The most important is that this operator has the right duality property with respect to the Malliavin divergence (see Proposition 10.1 below). Here is another and more “philosophical” explanation (assuming that L is a martingale): We may think of $x(\omega) = x(\omega_0, \omega_{\Delta t}, \dots, \omega_{N\Delta t})$ as a function of $N+1$ variables. If we use Taylor’s formula and formally replace $x(\omega_t^a)$ by its first order approximation $x(\omega) + D_t x(\omega) \cdot (a - \omega_t)$, we get (here \approx means “approximately equal” in an informal sense and not in the strict sense of nonstandard analysis):

$$\begin{aligned} Dx(\omega, t) &= \frac{1}{\sigma_L^2 \Delta t} \sum_{a \in A} x(\omega_t^a) a p_a \approx \\ &\approx \frac{1}{\sigma_L^2 \Delta t} \sum_{a \in A} (x(\omega) + D_t x(\omega)(a - \omega_t)) a p_a = D_t x(\omega) \end{aligned}$$

where we have used that $\sum_{a \in A} a p_a = 0$ (since L is a martingale) and that $\sigma_L^2 = \frac{1}{\Delta t} \sum_{a \in A} a^2 p_a$ by definition. This is the “intuitively correct” expression for the Malliavin gradient. In a recent paper [12], Di Nunno has a (standard) formula for the Malliavin gradient of Lévy random fields which is basically the same as our formula (11). Let me also add that it would be interesting to compare the theory developed here to Osswald’s abstract approach to Malliavin calculus on product spaces [31].

To study the duality between the Malliavin gradient and the Malliavin divergence, we need to introduce two internal L^2 -spaces. By $L^2(\Omega)$ we shall mean the space of all internal functions $x : \Omega \rightarrow {}^*\mathbf{R}$ with the norm

$$\|x\|_\Omega = E [|x|^2]^\frac{1}{2},$$

and by $L^2(\Omega \times T)$ we shall mean the set of all internal processes $X : \Omega \times T \rightarrow {}^*\mathbf{R}$ with the norm

$$\|X\|_{\Omega \times T} = E \left[\sum_{s < 1} |X(s)|^2 \sigma_L^2 \Delta t \right]^{\frac{1}{2}}$$

We use the corresponding notation for inner products in $L^2(\Omega)$ and $L^2(\Omega \times T)$. To make it easier for the reader, we shall denote elements in $L^2(\Omega)$ by lower case letters and elements in $L^2(\Omega \times T)$ by capital letters.

Note that we may regard D and δ as operators between our two L^2 -spaces:

$$D : L^2(\Omega) \rightarrow L^2(\Omega \times T)$$

$$\delta : L^2(\Omega \times T) \rightarrow L^2(\Omega)$$

Proposition 10.1 (Malliavin duality) *For all $x \in L^2(\Omega)$ and $Y \in L^2(\Omega \times T)$ we have*

$$\langle Dx, Y \rangle_{L^2(\Omega \times T)} = \langle x, \delta Y \rangle_{L^2(\Omega)}$$

Proof: Recall a simple property of conditional expectations: $E[fE[g|\mathcal{A}]] = E[E[f|\mathcal{A}]g]$. Using this we get:

$$\begin{aligned} \langle Dx, Y \rangle_{L^2(\Omega \times T)} &= E \left[\sum_{s < 1} \frac{1}{\sigma_L^2 \Delta t} E[x \Delta L(s) | \mathcal{G}_{\neq s}] Y(s) \sigma_L^2 \Delta t \right] = \\ &= E \left[\sum_{s < 1} E[Y(s) | \mathcal{G}_{\neq s}] x \Delta L(s) \right] = \\ &= E \left[x \cdot \sum_{s < 1} E[Y(s) | \mathcal{G}_{\neq s}] \Delta L(s) \right] = \langle x, \delta Y \rangle_{L^2(\Omega)} \quad \spadesuit \end{aligned}$$

Note that the proposition above would remain true if we removed σ_L^2 from both the definition of the Malliavin gradient D and from the norm of $L^2(\Omega \times T)$ (since these two occurrences cancel in the proof). The next result shows that the scaling introduced by σ_L^2 is natural.

Proposition 10.2 (Stochastic differentiation) *If L is a Lévy martingale and X is $\{\mathcal{G}_{<t}\}$ -adapted, then*

$$E[D(\delta X)(t) | \mathcal{G}_{<t}] = E[D(\int_0^1 X dL)(t) | \mathcal{G}_{<t}] = X(t)$$

Proof: We have

$$D(\int_0^1 X dL)(t) = \frac{1}{\sigma_L^2 \Delta t} E[(\sum_{s < 1} X(s) \Delta L(s)) \Delta L(t) | \mathcal{G}_{\neq t}] =$$

$$\begin{aligned}
&= \frac{1}{\sigma_L^2 \Delta t} \sum_{s < t} E[X(s) \Delta L(s) \Delta L(t) | \mathcal{G}_{\neq t}] + \frac{1}{\sigma_L^2 \Delta t} E[X(t) \Delta L(t)^2 | \mathcal{G}_{\neq t}] + \\
&\quad + \frac{1}{\sigma_L^2 \Delta t} \sum_{s > t} E[X(s) \Delta L(s) \Delta L(t) | \mathcal{G}_{\neq t}] = \\
&= 0 + X(t) + \frac{1}{\sigma_L^2 \Delta t} \sum_{s > t} E[X(s) \Delta L(s) \Delta L(t) | \mathcal{G}_{\neq t}]
\end{aligned}$$

where we have used that L is a martingale and that X is adapted to show that the first part of the sum is zero, and the adaptedness of X and the definition of σ_L to prove that the second term equals $X(t)$. To finish the proof, it suffices to observe that for $s > t$,

$$E[E[X(s) \Delta L(s) \Delta L(t) | \mathcal{G}_{\neq t}] | \mathcal{G}_{< t}] = E[X(s) \Delta L(s) \Delta L(t) | \mathcal{G}_{< t}] = 0$$

where we use that L is a martingale with independent increments and that X is adapted. ♠

It is easy to see that for general L , there will be internal random variables $x \in L^2(\Omega)$ which are not Itô integrals (in the terminology of mathematical finance, they are “claims” that can not be “hedged”). It is natural to ask what we get if we apply the operator in the proposition to such an x :

Proposition 10.3 (Clark-Haussmann-Ocone Formula) *Assume that L is a Lévy martingale and let E be the subspace of $L^2(\Omega)$ generated by the constants and all adapted Itô integrals, i.e.*

$$E = \{a + \int_0^1 Z \, dL : a \in {}^*\mathbf{R} \text{ and } Z \text{ is } \{\mathcal{G}_{< t}\}\text{-adapted}\}$$

The orthogonal projection $P_E : L^2(\Omega) \rightarrow E$ is given by

$$P_E(x) = E(x) + \int_0^1 E[Dx(t) | \mathcal{G}_{< t}] \, dL(t)$$

In other words, $E(x) + \int_0^1 E[Dx(t) | \mathcal{G}_{< t}] \, dL(t)$ is the stochastic integral closest to x in L^2 -norm.

Proof: Let

$$P(x) = E(x) + \int_0^1 E[Dx(t) | \mathcal{G}_{< t}] \, dL(t)$$

It suffices to show that

$$\langle x - Px, b + \int_0^1 Z \, dL \rangle_{L^2(\Omega)} = 0 \tag{12}$$

for all $b + \int_0^1 Z dL \in E$. We first observe that

$$\langle x, b + \int_0^1 Z dL \rangle_{L^2(\Omega)} = E(x)b + \sum_{s < 1} E[xZ(s)\Delta L(s)] \quad (13)$$

On the other hand

$$\begin{aligned} & \langle Px, b + \int_0^1 Z dL \rangle_{L^2(\Omega)} = \\ &= \langle E(x) + \sum_{t < 1} E[Dx(t)|\mathcal{G}_{<t}] \Delta L(t), b + \sum_{s < 1} Z(s) \Delta L(s) \rangle_{L^2(\Omega)} = \\ &= E(x)b + \sum_{s, t} E \left[\frac{1}{\sigma_L^2 \Delta t} E[x \Delta L(t) | \mathcal{G}_{<t}] Z(s) \Delta L(t) \Delta L(s) \right] = \\ &= E(x)b + \sum_{s \neq t} E \left[\frac{1}{\sigma_L^2 \Delta t} E[x \Delta L(t) | \mathcal{G}_{<t}] Z(s) \Delta L(t) \Delta L(s) \right] + \\ & \quad + \sum_s E \left[\frac{1}{\sigma_L^2 \Delta t} E[x \Delta L(s) | \mathcal{G}_{<s}] Z(s) \Delta L(s)^2 \right] = \\ &= E(x)b + 0 + \sum_{s < 1} E[E[x \Delta L(s) | \mathcal{G}_{<s}] Z(s)] = \\ &= E(x)b + \sum_{s < 1} E[xZ(s)\Delta L(s)] \end{aligned}$$

where we have repeatedly used the adaptedness of Z and the fact that L is a martingale with independent increments such that $E[\Delta L(s)^2] = \sigma_L^2 \Delta t$. Comparing the result to (13), we see that (12) is proved. ♠

Remark To call the formula above a “Clark-Haussmann-Ocone formula” may be to stretch terminology a bit as $E(x) + \int_0^1 E[Dx(t)|\mathcal{G}_{<t}] dL(t)$ will in general *not equal* the original random variable x , but just give the best possible approximation (when L is Anderson’s random walk, we do have equality as shown by Leitz-Martini [21]). In standard contexts, several papers (see, e.g., [9], [8]) have been devoted to showing how one may obtain versions of the *strict* Clark-Haussmann-Ocone formula in a Lévy market by adding more basic processes (in financial terms this means “completing the market” by adding new assets). In the hyperfinite setting, it is also possible to complete the market by enriching the timeline. We shall return to this question in a forthcoming paper [25].

Since this section is just meant as an appetizer, we have not proved any regularity properties of the Malliavin operators D and δ , and thus it may happen that our optimal hedging strategy $E[Dx(t)|\mathcal{G}_t]$ take infinite values or fluctuate widely between one point in T and the next. Which regularity requirements we should impose, depends on the interpretation of time in the model. We hope to return to this question in [25], but for the time being we just refer the

reader to the nonstandard papers on Malliavin calculus in gaussian models (see in particular Cutland and Ng [11] and Osswald [29], [30]) and mathematical finance (see in particular the book by Ng [27], the chapter on mathematical finance in Cutland's book [10], and the survey paper by Kopp [20]).

11 Looking ahead

I would have liked to end this paper with a couple of real applications, but the paper is already more than long enough, and the applications (with the exception of the brief glimpse of Malliavin calculus provided by the previous section) will have to wait for future papers. Let me, however, take the opportunity to point out four areas where I think the theory may be of use. The first area is mathematical finance where Lévy markets have recently obtained a lot of attention as natural extensions of diffusion based markets (see, e.g., the papers [5], [7], [8], [9], [22] and the book [35]). Discrete time financial models based on random walks are conceptually and technically much easier to handle than continuous time models, and hyperfinite Lévy processes may be a very useful tool in transferring insights from discrete to continuous models in the Lévy case. The second area I would like to point out is Malliavin calculus. There is an increasing interest in the Malliavin calculus of Lévy processes (see, e.g., [28], [22], [5], [13]) partly motivated by the needs of mathematical finance, and as shown in the previous section, hyperfinite Lévy processes seem well suited to deal with these questions. The third, and closely related topic, is white noise analysis which is also being extended to a Lévy setting (see, e.g., [14]) with financial applications in mind. A natural starting point would be to extend the discrete model based on Bernoulli random walks in [17] and [18] to a full nonstandard theory of white noise with respect to hyperfinite Lévy processes (see also [32]). An introduction to nonstandard methods in *gaussian* white noise analysis can be found in [27]. The fourth topic is of a different character. Random walk approximations play an important part in the theory of Lévy processes, but there are many situations where the analogy between Lévy processes and random walks has not yet been fully exploited (see Doney [15] for some interesting open problems in fluctuation theory). By combining random walks and Lévy processes in the same object, the theory of hyperfinite Lévy processes may facilitate the use of random walk techniques in the study of Lévy processes.

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